# MAT-61006 Introduction to Functional Analysis

## II Partial Exam 5. 12. 2016

### No notes, books, or calculator

**Problem 1.** Let $H$ be a Hilbert space. Define the following concepts.

(a) An orthonormal basis of $H$.

(b) The norm of an operator $T \in \mathcal{B}(H)$.

(c) The resolvent set and spectrum of an operator $T \in \mathcal{B}(H)$.

**Problem 2.** Let $H$ be a Hilbert space and $M \subset H$ a subspace.

(a) Define the orthogonal complement $M^\perp$ of $M$.

(b) Let $M$ and $N$ be subspaces. Show that $(M+N)^\perp = M^\perp \cap N^\perp$.
Here $M + N = \{ m + n \mid m \in M \text{ and } n \in N \}$.

**Problem 3.** Define the operator $T : L^2[0,\infty) \to L^2[0,\infty)$ by

$$(Tf)(t) = f(2t), \qquad t \geq 0, \quad f \in L^2[0,\infty).$$

Find the adjoint $T^*$ of $T$. The inner product in $L^2[0,\infty)$ is of course

$$\langle f, g \rangle = \int_0^\infty f(t)\overline{g(t)}\,\mathrm{d}t.$$

**Problem 4.** Let $H$ be a Hilbert space.

(a) Define the weak convergence of a sequence $(x_n)_{n=1}^\infty \subset H$.

(b) Show that the limit of a weakly convergent sequence is unique.

(c) Give an example of a weakly convergent sequence, which doesn't converge with respect to the norm.