TAMPERE UNIVERSITY OF TECHNOLOGY

MAT-60206 Mathematical analysis Exam, Sep 4, 2017	(Kauhanen, SC309)

No calculator. No written material.

1. (a) Let $f\colon \mathbb{R} \to \mathbb{R}$. Give the exact definition for $\lim_{x\to\infty} f(x) = \infty$.

(b) Suppose that f and $g\colon \mathbb{R} \to \mathbb{R}$ are functions such that $0 < 1/f(x) < 2$ for all $x \geq 7$, and

$$\lim_{x\to\infty} g(x) = \infty.$$

Use your definition to prove that $\lim_{x\to\infty} (f(x)g(x)) = \infty$.

2. (a) Let $f\colon \mathbb{R} \to \mathbb{R} \setminus \{0\}$ be differentiable at $x_0 \in \mathbb{R}$. Use the definition of the derivative to prove that $1/f$ is differentiable at x_0 and to calculate $(1/f)'(x_0)$.

(b) Suppose that $g\colon \mathbb{R} \to \mathbb{R}$ is differentiable and that there is $c \geq 0$ such that $g'(x) \geq c$ for all $x \in \mathbb{R}$. Prove that for all $x \geq 0$,

$$g(x) \geq g(0) + cx.$$

3. Let f and $g\colon [a,b] \to \mathbb{R}$ be bounded functions such that $g(x) \leq f(x)$ for all $x \in [a,b]$.

(a) Let P be a partition of $[a,b]$. Prove that $s_g(P) \leq s_f(P)$.

(b) Using the definition and part (a), prove that $\underline{\int_a^b} g \leq \underline{\int_a^b} f$.

4. (a) Let

$$f_n(x) = \begin{cases} 0, & x = 0, \\ n, & 0 < x < 1/n, \\ 0, & 1/n \leq x \leq 1. \end{cases}$$

Show that $\lim_{n\to\infty} \int_0^1 f_n \neq \int_0^1 \lim_{n\to\infty} f_n$.

(b) Let $g_n(x) = \dfrac{1}{1+x^{2n}}$. Find the limit $\lim_{n\to\infty} \int_{-1/2}^{1/2} g_n$.

Käännä!