Hint: use the rotation $R(0,\theta,\varphi)$, the functional rotation for $R(\gamma,\beta,\alpha)$

$$Y_l'^m(\vartheta,\psi) = \sum_{m'=-l}^{l} Y_l^{m'}(\vartheta,\psi)\, D^{(l)}_{m'm}(\gamma,\beta,\alpha), \tag{4}$$

where the element $D^{(l)}_{m'm}$ of a rotation matrix $D^{(l)}$ is

$$D^{(l)}_{m'm}(\gamma,\beta,\alpha) = e^{im'\gamma}\, d^{(l)}_{m'm}(\beta)\, e^{im\alpha}, \tag{5}$$

and the identity

$$Y_l^m(\vartheta,\psi) \equiv D^{(l)}_{0m}(0,\vartheta,\psi). \tag{6}$$

5. Consider a Markov chain process with two states A and B, and the probability of the transition $A \rightarrow B$ given by a and $B \rightarrow A$ by b:

$$_{1-a} \circlearrowleft A \;_a \longrightarrow \longleftarrow_b\; B \circlearrowright_{1-b} \tag{7}$$

where $0 \le a, b \le 1$.

a) Write the transition matrix M of the system, and find the stationary population vector $\hat{f} \in \mathbb{R}^2$ such that $M\hat{f} = \hat{f}$. For all population vectors $f = (f_A, f_B)$, $f_A + f_B = 1$. Is there a periodic limit cycle such that the initial $f^{(0)}$ does not evolve towards $\hat{f}$?

b) In an East European country, formerly with a 100% vote for Communists, election results in 1990 were 50% for Democrats, 50% for Communists. 1994 the count was 62.5% (D), 37.5% (C). What is the count likely to be in this century, assuming a Markovian voting population and a continued two-party political stage? In another country (apparently populated by rather disloyal and cynical people), the respective counts were 1990: 90% (D), 10% (C); 1994: 18% (D), 82% (C). What is the equilibrium count going to be?

c) Analyze the convergence of the chain $f^{(n+1)} = Mf^{(n)}$ towards $\hat{f}$ from any initial $f^{(0)}$: by finding the eigenvalues λ_1, λ_2 from $\det(M - \lambda\mathbb{I}) = 0$ ($\mathbb{I}$ identity matrix) and writing the usual diagonal form $M = PDP^{-1}$, $D = Diag(\lambda_i)$ (do you need to construct the eigenvector matrix P here?), show that n successive multiplications of M can be given in the form

$$M^n = PD^nP^{-1} = F_1 + \lambda_2^n F_2, \tag{8}$$

and show that hence $F_1 f = \hat{f}$ must hold for all f. What happens to the F_2-term with increasing n? How is this related to the Markov chain Monte Carlo "burn-in" time? With which values of a, b does the system convergence to f in one step from any $f^{(0)}$? Which values of a, b make the sign of the F_2-term alternate between successive Markov steps?

That's it for this year. Merry Xmas and Happy New Year!