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Inverse Problems, MAT-52506 and Inversio-ongelmat, MAT-52500

Course exam 16 Dec 2009

Please answer in Finnish or in English. Choose any **four** of the following five problems.

1. Write and discuss an integral equation (N dimensions) describing a typical inverse problem. How do you discretize the problem such that you have a finite number of parameters to determine? Consider the case when the equation is linear in terms of these parameters. In what ways can you study the uniqueness, stability, and information content of the solution of the inverse problem (analytically and numerically)?

2. Consider the case of a strongly ill-posed inverse problem and some available a priori information or assumptions. How do you incorporate the a priori component in the problem? How do you determine the weight of a regularizing function (e.g., of Tikhonov type)? What if you have another complementary data source rather than a prior constraint: how do you combine two data modalities, and what is the corresponding maximum compatibility estimate?

3. Explain the Bayesian principle and discuss its use in inverse problems. Assume a linear problem with a Gaussian (normal) error distribution, and a Gaussian distribution representing the prior assumptions on the model. Derive the equations for the posterior distribution and its central point in terms of the Fisher information matrix

$$Q = (\Sigma_0^{-1} + A^T \Sigma_1^{-1} A), \tag{1}$$

where Σ_0 and Σ_1 are related to the a priori and the linear model distributions, and A is the model matrix.

4. Assume we have a complete set of radar echo power measurements $L(\omega, d)$, $0 \leq \mathrm{d} \leq 1$, $\omega = (\theta, \varphi) \in S^2$ of a unit sphere (d related to the distance of the radar from a point on the sphere) such that they can be given in the $L^2(S^2)$-form

$$L(\omega, d) = \sum_{lm} a_{lm}(d) Y_l^m(\omega), \tag{2}$$

where

$$Y_l^m(\omega) = P_l^m(\cos\theta) e^{im\varphi}, \tag{3}$$

and P_l^m are associated Legendre polynomials. Suppose that the sphere has a coating of material whose reflectivity $\rho(\omega)$ can be described as an $L^2(S^2)$-function, and whose scattering behaviour $S(\mu)$, $\mu = \langle \omega, \nu \rangle$ (ν the unit surface normal), can be given as a polynomial. Show that these data uniquely determine $\rho(\omega)$. Can $S(\mu)$ be determined as well? Is the problem apparently ill-posed? How do the uniqueness and stability differ from the case of data integrated over d (i.e., measurements of integrated brightness only)?